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# The Green function of a perturbed network

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**Abstract.** The discrete Green functions and their relationship with discrete Laplace equations have deserved the interest of many researchers using different approaches. In this work we derive the Green function of a perturbed network in terms of the Green function of its base network.

**Key words:** Perturbed Laplacian, Green function, Effective Resistance, Lattice.

## 1 Introduction

Given a finite set  $V$ , whose cardinality equals  $n$ , the set of real valued functions on  $V$  is denoted by  $\mathcal{C}(V)$ . In particular, for any  $x \in V$ ,  $\varepsilon_x$  denotes the Dirac function at vertex  $x$ . An endomorphism  $\mathcal{K}$  of  $\mathcal{C}(V)$  is called *positive semi-definite* when  $\langle \mathcal{K}(u), u \rangle \geq 0$  for any  $u \in \mathcal{C}(V)$  and *positive definite* when  $\langle \mathcal{K}(u), u \rangle > 0$  for any non-null  $u \in \mathcal{C}(V)$ . A self-adjoint operator  $\mathcal{K}$  is named *elliptic* if it is positive semi-definite and its rank is greater than or equal to  $n - 1$ .

A function  $K: V \times V \rightarrow \mathbb{R}$  is generically called *a kernel on  $V$*  and determines an endomorphism of  $\mathcal{C}(V)$  by assigning to any  $u \in \mathcal{C}(V)$  the function  $\mathcal{K}(u) = \sum_{y \in V} K(\cdot, y) u(y)$ . Conversely, each endomorphism of  $\mathcal{C}(V)$  is deter-

mined by the kernel given for any  $x, y \in V$  by  $K(x, y) = \langle \mathcal{K}(\varepsilon_y), \varepsilon_x \rangle$ . Therefore, an endomorphism  $\mathcal{K}$  is self-adjoint iff its kernel is a symmetric function. Given  $\sigma, \tau \in \mathcal{C}(V)$  we denote by  $\mathcal{P}_{\sigma, \tau}$  the endomorphism whose kernel is  $\sigma \otimes \tau$ . In particular, when  $\tau = \sigma$  the above endomorphism is denoted simply by  $\mathcal{P}_\sigma$ .

If we consider an arbitrary order of the elements of  $V$ , then kernels, and hence its associated endomorphisms of  $\mathcal{C}(V)$ , can be identified with matrices of order  $n$ , whereas functions on  $V$  can be alternatively identified with (column)

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vectors of  $\mathbb{R}^n$  or diagonal matrices. In particular, if  $\mathcal{I}$  denotes the Identity on  $\mathcal{C}(V)$ , it is identify with the Identity matrix  $I$ . On the other hand, an endomorphism  $\mathcal{K}$  of  $\mathcal{C}(V)$  is identified with a symmetric or positive semi-definite matrix iff  $\mathcal{K}$  has the same property, see [?].

## 2 Perturbed networks

In this section we consider fixed the connected finite network  $\Gamma$  on the vertex set  $V$  whose conductance is the symmetric kernel  $c: V \times V \rightarrow [0, +\infty)$  which satisfies that  $c(x, x) = 0$  for any  $x \in V$  and moreover, vertex  $x$  is adjacent to vertex  $y$  iff  $c(x, y) > 0$ . We call  $\Gamma$  the *base network*.

The *combinatorial Laplacian* or simply the *Laplacian* of the network  $\Gamma$  is the endomorphism of  $\mathcal{C}(V)$  that assigns to each  $u \in \mathcal{C}(V)$  the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) (u(x) - u(y)), \quad x \in V. \quad (1)$$

It is well-known, that the Laplacian is a singular self-adjoint and positive semi definite on  $\mathcal{C}(V)$  and moreover  $\mathcal{L}(u) = 0$  iff  $u$  is a constant function. We denote by  $\mathcal{G}$  and  $G$  the orthogonal Green operator and the Green function, respectively. It is well-known that  $\mathcal{G}$  is singular self-adjoint and positive semi definite on  $\mathcal{C}(V)$  and moreover  $\mathcal{G}(u) = 0$  iff  $u$  is a constant function

Our next aim is to analyze the Laplacian operators associated with a perturbation of the conductance. Specifically we consider  $\epsilon: V \times V \rightarrow [0, +\infty)$  a symmetric function and denote by  $\mathcal{L}^\epsilon$  the combinatorial laplacian associated with the perturbed conductance  $c + \epsilon$  and by  $\mathcal{G}^\epsilon$  and  $G^\epsilon$  the corresponding orthogonal Green operator and Green function, respectively.

For any  $x, y \in V$  we consider  $\sigma_{xy} = \sqrt{\frac{\epsilon(x, y)}{2}}(\varepsilon_x - \varepsilon_y)$  and  $R(x, y)$  the *dipole* and the *effective resistance* between  $x$  and  $y$ , respectively. Effective resistance and Green function are related by  $R(x, y) = G(x, x) + G(y, y) - 2G(x, y)$ , which implies that  $R: V \times V \rightarrow [0, +\infty)$  is symmetric and vanishes on the diagonal, see [?] and references therein. We can extend the effective resistance to  $R: V \times V \times V \times V \rightarrow [0, +\infty)$  by

$$R(x, y, z, t) = \frac{1}{2} \left( R(x, t) - R(t, y) + R(y, z) - R(z, x) \right).$$

since  $R(x, y, x, y) = R(x, y)$ .

**Theorem 1.** *If is satisfied that*

$$\mathcal{L}^\epsilon = \mathcal{L} + \sum_{x, y \in V} \mathcal{P}_{\sigma_{xy}} \quad \text{and} \quad \mathcal{G}^\epsilon = \mathcal{G} - \sum_{x, y, z, t \in V} b(xy, zt) \mathcal{P}_{\mathcal{G}(\sigma_{xy}), \mathcal{G}(\sigma_{zt})},$$

where  $(b(xy, zt)) = \left[ 1 + (\langle \mathcal{G}(\sigma_{xy}), \sigma_{zt} \rangle) \right]^{-1}$ .

*Proof.* For the first equality, it suffices to observe that given  $u \in \mathcal{C}(V)$ , then for all  $z \in V$ ,

$$\begin{aligned} \sum_{x,y \in V} \mathcal{P}_{\sigma_{xy}}(u)(z) &= \frac{1}{2} \sum_{x,y \in V} \epsilon(x,y)(\varepsilon_x(z) - \varepsilon_y(z))(u(x) - u(y)) \\ &= \frac{1}{2} \sum_{y \in V} \epsilon(z,y)(u(z) - u(y)) - \frac{1}{2} \sum_{x \in V} \epsilon(x,z)(u(x) - u(z)) \\ &= \frac{1}{2} \sum_{y \in V} \epsilon(z,y)(u(z) - u(y)) + \frac{1}{2} \sum_{y \in V} \epsilon(z,y)(u(z) - u(y)) \\ &= \sum_{y \in V} \epsilon(z,y)(u(z) - u(y)). \end{aligned}$$

On the other hand, the Poisson equation  $\mathcal{L}^\epsilon(u) = f$  where  $f \in \mathcal{C}(V)$  and  $\langle f, \mathbf{1} \rangle = 0$  is equivalent to the Poisson equation

$$\mathcal{L}(u) = f - \sum_{x,y \in V} \mathcal{P}_{\sigma_{xy}}(u) = f - \sum_{x,y \in V} \langle \sigma_{xy}, u \rangle \sigma_{xy}.$$

Therefore, the unique solution of  $\mathcal{L}^\epsilon(u) = f$  such that  $\langle u, \mathbf{1} \rangle = 0$  is given by

$$u = \mathcal{G}(f) - \sum_{x,y \in V} \langle \sigma_{xy}, u \rangle \mathcal{G}(\sigma_{xy})$$

and multiplying by  $\sigma_{zt}$ , we get that

$$\langle \sigma_{zt}, u \rangle + \sum_{x,y \in V} \langle \sigma_{xy}, u \rangle \langle \mathcal{G}(\sigma_{xy}), \sigma_{zt} \rangle = \langle \sigma_{zt}, \mathcal{G}(f) \rangle = \langle \mathcal{G}(\sigma_{zt}), f \rangle.$$

The coefficient matrix of the above system is  $\mathbf{I} + (\langle \mathcal{G}(\sigma_{xy}), \sigma_{zt} \rangle)$  and it is non singular, because it is symmetric and positive definite. Therefore,

$$\langle \sigma_{xy}, u \rangle = \sum_{z,t \in V} b(xy, zt) \langle \mathcal{G}(\sigma_{zt}), f \rangle,$$

and then  $u = \mathcal{G}(f) - \sum_{x,y,z,t \in V} b(xy, zt) \mathcal{G}(\sigma_{xy}) \langle \mathcal{G}(\sigma_{zt}), f \rangle$ . □

**Corollary 1.** *If  $R^\epsilon$  is the effective resistance on the perturbed network, given  $v, w \in V$  then*

$$R^\epsilon(v, w) = R(v, w) - \frac{1}{2} \sum_{x,y,z,t \in V} b(xy, zt) \sqrt{\epsilon(x,y)\epsilon(z,t)} R(x, y, v, w) R(z, t, v, w).$$

*In particular, when  $\epsilon(u, w) > 0$ , then  $R^\epsilon(v, w) = \frac{2(1 - b(vw, vw))}{\epsilon(v, w)}$ .*

*Proof.* From the identity  $\mathcal{G}(\sigma_{xy}) = \sqrt{\frac{\epsilon(x,y)}{2}}(G(\cdot, x) - G(\cdot, y))$  we obtain

$$\langle \mathcal{G}(\sigma_{xy}), \sigma_{zt} \rangle = \frac{\sqrt{\epsilon(x,y)\epsilon(z,t)}}{2}(G(z, x) - G(z, y) + G(t, y) - G(t, x))$$

and hence, the relation between Green function and effective resistance implies that

$$\langle \mathcal{G}(\sigma_{xy}), \sigma_{zt} \rangle = \frac{\sqrt{\epsilon(x,y)\epsilon(z,t)}}{2}R(x, y, z, t).$$

Therefore, the first identity follows from the above theorem applying newly the relation between effective resistances and Green functions.

On the other hand, when  $\epsilon(u, v) > 0$ , the identity can be expressed as

$$R^\epsilon(v, w) = R(v, w) - \frac{2}{\epsilon(v, w)} \sum_{x, y, z, t \in V} b(xy, zt) \langle \mathcal{G}(\sigma_{xy}), \sigma_{vw} \rangle \langle \mathcal{G}(\sigma_{zt}), \sigma_{vw} \rangle.$$

If  $\Lambda = (\langle \mathcal{G}(\sigma_{xy}), \sigma_{zt} \rangle)$ , then  $\Lambda B \Lambda = \Lambda + B - I$ , because  $B = (I + \Lambda)^{-1}$ , and hence

$$\sum_{x, y, z, t \in V} b(xy, zt) \langle \mathcal{G}(\sigma_{xy}), \sigma_{vw} \rangle \langle \mathcal{G}(\sigma_{zt}), \sigma_{vw} \rangle = \langle \mathcal{G}(\sigma_{vw}), \sigma_{vw} \rangle + b(vw, vw) - 1.$$

The last claim follows from the identity  $\langle \mathcal{G}(\sigma_{vw}), \sigma_{vw} \rangle = \frac{\epsilon(v, w)}{2}R(v, w)$ .  $\square$

**Corollary 2.** *The Green operator on a connected network is a perturbation of the Green operator of any spanning tree of the network. In particular if the given network has a Hamiltonian path, then the Green operator is a perturbation of the Green operator of the Hamiltonian path.*

In practice, it is more useful to describe the above results in terms of edges instead of vertices and to consider only those added edges with positive conductance. Next, we exemplify this observation describing the simple case in which the perturbation is due to the addition of a single edge.

**Corollary 3.** *Assume that we add an edge of conductance  $\epsilon > 0$  between vertices  $x$  and  $y$  and consider the dipole  $\sigma = \sqrt{\epsilon}(\varepsilon_x - \varepsilon_y)$ . Then,*

$$\mathcal{G}^\epsilon = \mathcal{G} - \frac{1}{1 + \epsilon R(x, y)} \mathcal{P}_{\mathcal{G}(\sigma)}.$$

Therefore, for all  $v, w \in V$  it is satisfied

$$R^\epsilon(v, w) = R(v, w) - \frac{\epsilon R(x, y, v, w)^2}{1 + \epsilon R(x, y)}$$

and, in particular,  $R^\epsilon(x, y) = \frac{1}{\epsilon + \frac{1}{R(x, y)}}$ .

We finish this section noticing that the result in Theorem ?? can be obtained applying recursively the above Corollary according with the number,  $k$ , of added edges. Specifically, if for any  $i = 1, \dots, k$  we consider  $\sigma_i$ , the dipole corresponding to the  $i$ -th added edge,  $\mathcal{L}_i = \mathcal{L} + \sum_{j=1}^i \mathcal{P}_{\sigma_j}$  the combinatorial Laplacian and  $\mathcal{G}_i$  its Green operator, then

$$\mathcal{G}_i = \mathcal{G}_{i-1} - \frac{1}{1 + \langle \mathcal{G}_{i-1}(\sigma_i), \sigma_i \rangle} \mathcal{P}_{\mathcal{G}_{i-1}(\sigma_i)},$$

because  $\mathcal{L}_i = \mathcal{L}_{i-1} + \mathcal{P}_{\sigma_i}$ ,  $i = 1, \dots, k$ , where  $\mathcal{G}_0$  denotes the Green operator for  $\mathcal{L}_0 = \mathcal{L}$ .

This strategy recovers the method used to obtain the Moore-Penrose inverse of perturbed irreducible tridiagonal matrices, see [?] for the nonsingular case and [?] for singular matrices.

### 3 Application: The Green function of a lattice

In this section we apply the above techniques to determine the Green function of the  $3 \times 3$ -lattice whose vertices are labeled  $V = \{x_1, \dots, x_9\}$ . The conductance function is defined as follows, see Figure ??:

- (i)  $c(x_i, x_{i+1}) = c > 0$ , for  $i = 1, \dots, 8$ .
- (ii)  $c(x_1, x_6) = c(x_2, x_5) = c(x_5, x_8) = c(x_4, x_9) = a > 0$ .

**Fig. 1.**  $3 \times 3$ -lattice.

The Hamiltonian path has constant conductance equal to  $c$  and its Green function, is given by, see [?]

$$G(x_i, x_j) = \frac{1}{54c} \left[ 190 + 3[i(i-10) + j(j-10) - 9|i-j|] \right], \quad i, j = 1, \dots, 9,$$

which implies that the effective resistance is  $R(x_i, x_j) = \frac{|i-j|}{c}$ ,  $i, j = 1, \dots, 9$ .

The matrix associated to  $\mathcal{G}$  is

$$\mathbf{G} = \frac{1}{27c} \begin{bmatrix} 68 & 44 & 23 & 5 & -10 & -22 & -31 & -37 & -40 \\ 44 & 47 & 26 & 8 & -7 & -19 & -28 & -34 & -37 \\ 23 & 26 & 32 & 14 & -1 & -13 & -22 & -28 & -31 \\ 5 & 8 & 14 & 23 & 8 & -4 & -13 & -19 & -22 \\ -10 & -7 & -1 & 8 & 20 & 8 & -1 & -7 & -10 \\ -22 & -19 & -13 & -4 & 8 & 23 & 14 & 8 & 5 \\ -31 & -28 & -22 & -13 & -1 & 14 & 32 & 26 & 23 \\ -37 & -34 & -28 & -19 & -7 & 8 & 26 & 47 & 44 \\ -40 & -37 & -31 & -22 & -10 & 5 & 23 & 44 & 68 \end{bmatrix}$$

The four added edges have the same conductance  $a$ . We denote by  $\sigma_{16}, \sigma_{25}, \sigma_{58}$  and  $\sigma_{49}$  the corresponding dipoles and moreover,

$$\mathbb{1} + (\langle \mathcal{G}(\sigma_{xy}), \sigma_{zt} \rangle) = \frac{1}{c} \begin{bmatrix} c+5a & 3a & a & 2a \\ 3a & c+3a & 0 & a \\ a & 0 & c+3a & 3a \\ 2a & a & 3a & c+5a \end{bmatrix}.$$

Although our method to obtain the Green function of a complex network throughout the Green function of a base network, works in the general setting, for the sake of simplicity in the sequel we assume  $a = c$ , which in particular implies that the lattice appears as Cartesian product of two paths with length three.

$$\left[ \mathbb{1} + (\langle \mathcal{G}(\sigma_{xy}), \sigma_{zt} \rangle) \right]^{-1} = \frac{1}{24} \begin{bmatrix} 7 & -5 & -1 & -1 \\ -5 & 10 & 2 & -1 \\ -1 & 2 & 10 & -5 \\ -1 & -1 & -5 & 7 \end{bmatrix}$$

From Theorem ??, we obtain the Green function of the lattice

$$\mathbf{G}^c = \frac{1}{72c} \begin{bmatrix} 39 & 7 & -6 & -11 & -4 & 7 & -6 & -11 & -15 \\ 7 & 26 & 7 & -4 & 0 & -4 & -11 & -10 & -11 \\ -6 & 7 & 39 & 7 & -4 & -11 & -15 & -11 & -6 \\ -11 & -4 & 7 & 26 & 0 & -10 & -11 & -4 & 7 \\ -4 & 0 & -4 & 0 & 16 & 0 & -4 & 0 & -4 \\ 7 & -4 & -11 & -10 & 0 & 26 & 7 & -4 & -11 \\ -6 & -11 & -15 & -11 & -4 & 7 & 39 & 7 & -6 \\ -11 & -10 & -11 & -4 & 0 & -4 & 7 & 26 & 7 \\ -15 & -11 & -6 & 7 & -4 & -11 & -6 & 7 & 39 \end{bmatrix}.$$

Next, we obtain the above Green function iteratively, by adding an edge at each time. In each step the obtained function is the Green function of a new network.

We start adding to the path the edge with dipole  $\sigma_{16}$ , see Figure ?? left. Then,

**Fig. 2.** Addition of dipoles  $\sigma_{16}$  (left) and  $\sigma_{25}$  (right).

$$\mathbf{G}_1 = \frac{1}{18c} \begin{bmatrix} 12 & 6 & 2 & 0 & 0 & 2 & -4 & -8 & -10 \\ 6 & 15 & 8 & 3 & 0 & -1 & -7 & -11 & -13 \\ 2 & 8 & 16 & 8 & 2 & -2 & -8 & -12 & -14 \\ 0 & 3 & 8 & 15 & 6 & -1 & -7 & -11 & -13 \\ 0 & 0 & 2 & 6 & 12 & 2 & -4 & -8 & -10 \\ 2 & -1 & -2 & -1 & 2 & 7 & 1 & -3 & -5 \\ -4 & -7 & -8 & -7 & -4 & 1 & 13 & 9 & 7 \\ -8 & -11 & -12 & -11 & -8 & -3 & 9 & 23 & 21 \\ -10 & -13 & -14 & -13 & -10 & -5 & 7 & 21 & 37 \end{bmatrix}$$

Adding the edge with dipole  $\sigma_{25}$ , see Figure ?? right we obtain

$$\mathbf{G}_2 = \frac{1}{45c} \begin{bmatrix} 28 & 10 & 3 & 1 & 4 & 6 & -9 & -19 & -24 \\ 10 & 25 & 15 & 10 & 10 & 0 & -15 & -25 & -30 \\ 3 & 15 & 38 & 21 & 9 & -4 & -19 & -29 & -34 \\ 1 & 10 & 21 & 37 & 13 & -3 & -18 & -28 & -33 \\ 4 & 10 & 9 & 13 & 22 & 3 & -12 & -22 & -27 \\ 6 & 0 & -4 & -3 & 3 & 17 & 2 & -8 & -13 \\ -9 & -15 & -19 & -18 & -12 & 2 & 32 & 22 & 17 \\ -19 & -25 & -29 & -28 & -22 & -8 & 22 & 57 & 52 \\ -24 & -30 & -34 & -33 & -27 & -13 & 17 & 52 & 92 \end{bmatrix}$$

Adding the edge with dipole  $\sigma_{58}$ , see Figure ?? left we obtain

**Fig. 3.** Addition of dipoles  $\sigma_{58}$  (left) and  $\sigma_{49}$  (right).

$$\mathbf{G}_3 = \frac{1}{1512c} \begin{bmatrix} 835 & 175 & -74 & -155 & -68 & 151 & -146 & -275 & -443 \\ 175 & 595 & 238 & 49 & 28 & -77 & -266 & -287 & -455 \\ -74 & 238 & 988 & 394 & -32 & -218 & -380 & -374 & -542 \\ -155 & 49 & 394 & 907 & 76 & -191 & -326 & -293 & -461 \\ -68 & 28 & -32 & 76 & 352 & 4 & -104 & -44 & -212 \\ 151 & -77 & -218 & -191 & 4 & 547 & 142 & -95 & -263 \\ -146 & -266 & -380 & -326 & -104 & 142 & 844 & 202 & 34 \\ -275 & -287 & -374 & -293 & -44 & -95 & 202 & 667 & 499 \\ -443 & -455 & -542 & -461 & -212 & -263 & 34 & 499 & 1843 \end{bmatrix}$$

Finally the addition of the edge with dipole  $\sigma_{49}$ , see Figure ?? right, leads to the previously obtained Green function of the lattice,  $\mathbf{G}^\epsilon$ , as expected.

To conclude, we notice that the lattice we have been working with, can be seen as the Cartesian product of  $P_3$ , the path of length 3 with constant conductance. We also recall that the Green function of a network  $\Gamma$ , can be expressed in terms of its Laplacian orthonormal eigensystem  $\{(\lambda_i, \Phi_i), i \in I\}$ , as

$G = \sum_{\lambda_i > 0} \frac{1}{\lambda_i} \Phi_i \Phi_i^\top$ , where  $\Phi_i$  is the normalized eigenfunction of  $\lambda_i$ , a Laplacian eigenvalue of  $\Gamma$ .

Moreover, it is well known that if  $\Gamma$  and  $\Gamma'$  have eigensystems  $\{(\lambda_i, \Phi_i), i \in I\}$ , and  $\{(\mu_j, \Psi_j), j \in J\}$  resp., then the eigensystem of the Cartesian product is  $\{\sigma_k, \Lambda_k, k \in I \cup J\}$ , where  $\sigma_k = \lambda_i + \lambda_j$ ,  $\Lambda_k = \Phi_i \otimes \Psi_j$  and  $\otimes$  is the Kronecker product.

According to this results and taking into account that  $P_3$  has eigensystem  $\{(0, \frac{1}{\sqrt{3}}(1, 1, 1)); (1, \frac{1}{\sqrt{2}}(1, 0, -1)); (3, \frac{1}{\sqrt{2}}(1, 0, -1))\}$ , we get another way to derive the Green function of the lattice.

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